Equations (2.24) with inhomogeneous terms (3.11) do not contain any non-equilibrium parameters which characterize only the gaseous systems. Therefore they can be used to study non-equilibrium, large-scale fluctuations in fluid flows.

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ASYMPTOTIC FORM OF SMALL DENSITY DIFFERENCES IN THE PROBLEM OF COHERENT PHASE TRANSFORMATIONS*

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Equations describing (in the lower approximation) the equilibrium configurations under heterogeneous, coherent phase transformations in an elastic, one-component medium, are derived for the asymptotic case of small density differences. Both phases are assumed to be isotropic by virtue of the multiplicity and certain computational simplifications. It is shown that, to a first approximation, the equilibrium temperature of the non-hydrostatic, two-phase configuration is identical with the temperature of phase equilibrium of the hydrostatically stressed phases in some reference configuration. In a higher approximation the system of equations of equilibrium obtained is identical with the equations of the classical linear theory of elasticity, although, on the whole, the problem remains essentially non-linear, since it contains an unknown boundary and certain boundary conditions on it, quadratic with respect to the displacement. The conditions obtained are further used to find the solutions of certain boundary value problems.

The conditions of equilibrium obtained in /1, 2/ under coherent phase transformations with slippage, represent special boundary value problems for the equations of the non-linear theory of elasticity, with unknown boundaries. The presence of unknown boundaries of contact between the different phases aggravates the difficulties of the already complicated problem of describing the equilibrium configurations of non-linearly elastic materials (e.g. in the simplest problem of this type for a liquid system where the problem reduces to that of determining the equilibrium values of the pressures, temperture and phase masses, the equilibrium

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conditions degenerate into a complex, non-linear algebraic system). For this reason, the asymptotic form of a small difference in the phase densities, developed in the present paper using the case of coherent phase transformations in a medium that is isotropic (in both phase states) is of considerable interest.

1. The conditions of equilibrium under coherent phase transformations in a simple, elastic, one-component material. Following /1/ we shall carry out the investigation using the Lagrangian variables x^i and transforming somewhat, for convenience, the relations obtained in /1/. We shall distinguish between two isotropic elastic phases by labelling them with a plus and minus sign. Let us consider a homogeneous, elastic material in the plus phase state, and assume that the Lagrangian x^i coordinate system is affine in the reference configuration in question, with the basis $\mathbf{x}_{i+}$. Let us next consider a second homogeneous reference configuration of the same material, in the minus phase state, and assume that the passage from the first configuration to the second is accompanied by the corresponding volume expansion-compression deformation with similarity factor d, so that the corresponding displacement field $\mathbf{w}(x)$ of the material point with coordinates x^i is given by the formula $\mathbf{w} = (d-1)x^i\mathbf{x}_{i+}$; the basis of this configuration is given by $\mathbf{x}_{i-} = d\mathbf{x}_{i+}$.

Let us consider, in x^i coordinate space, the surface $\xi - x^i = x^i(\xi^\alpha), \alpha = 1, 2$. According to /1/ the problem of describing the equilibrium in the case of coherent phase transformations reduces to determining the equilibrium temperture θ, the unknown boundary ξ of the plus phase displacement field $\mathbf{u}_+(x)$ on one side of ξ, and of the field of (additional) displacement of the minus phase $\mathbf{u}_-(x)$ on the other side, so that the conditions a) and b) - d) would hold within the phases and on the interphase boundary respectively

$$\text{a) } p^{ji}_{,j} = 0, \quad \text{b) } [U^i]_-^+ = 0, \quad \text{c) } [p^{ji}]_-^+ n_j = 0, \quad \text{d) } [v^{ji}]_-^+ n_j n_i = 0 \tag{1.1}$$

Here $p_\pm^{ji}$ is the Piola-Kirchhoff stress tensor referred (for both phases) to the reference configuration of the plus phase; the index following the coma denotes partial differentiation, which in this case is identical with covariant differentiation, and $U_\pm^i$ are the components of the total displacement fields in the basis $\mathbf{x}_{i+}$ ($\mathbf{U}_+ = \mathbf{u}_+, \mathbf{U}_- = \mathbf{w} + \mathbf{u}_-$). We further have

$$v_\pm{}^{ij} = \psi x_+{}^{ij} - \frac{1}{m_+} p_\pm^{ik} U_{k,l\pm} x_+{}^{lj} \tag{1.2}$$

where $\psi_\pm$ denote the free energy densities of the phases per unit mass and $x_{ij\pm}$ $(x_\pm{}^{ij})$, $m_\pm$ are metric tensors and mass densities of the phases in the reference configurations. We denote by n_j the components of the unit normal to the image of the surface ξ in the reference configuration of the plus phase. Using the metric volume and surface tensors corresponding to this configuration, we carry out covariant differentiation and "juggle" the indices (unless the contrary is clearly indicated).

To close system (1.1) (despite the condition that the absolute temperature is constant and the specifying of the particular form of the function ψ), we must also specify the conditions either on the outer boundary of the body, or at infinity.

Let us denote by $u_{i-}{}^*$ the components of the field $\mathbf{u}_-$ in the basis $\mathbf{x}_{i-}$. The free energy ψ of the minus phase will conveniently be specified in what follows as a function of the absolute temperature θ and of the displacement gradients $u^*_{i,j}$. However, the derivatives of this function are connected with the tensor $p_-{}^{ji}$ introduced by projecting the stress tensors with vector components $\mathbf{P}_-^j$ /3/ onto the basis $\mathbf{x}_{i+}$, in a vert complicated manner. To overcome this difficulty, we shall introduce another Piola-Kirchhoff stress tensor p_-^{*ji} by expanding $\mathbf{P}_\pm^{*j}$ over the basis of the reference configuration of the minus phase x_{i-}. The tensors $p_\pm^{*ji}$ are connected with the free energy densities $\psi(u^*_{k,l\pm}, \theta)$ by the usual relations (here and henceforth we shall write, in order to save space, the similar formulas simultaneously for both phases, with the asterisk indicating the minus phase only)

$$p_\pm^{*ji} = m_\pm \partial\psi(u^*_{k,l\pm}, \theta)/\partial u^*_{i,j\pm}$$

Standard geometrical relations lead to the following relation connecting the stress tensors: $p_-^{ji} = p_-^{*ji} m_+ d\, m_-$.

Expanding $\psi_\pm, p_\pm^{*ji}$ in series in arguments $u^*_{k,l\pm}$ and $T = \theta - \theta^\circ$, we obtain

$$\begin{aligned} \psi_\pm &= \psi_\pm{}^\circ + \psi_{\theta\pm} T + \psi_\pm{}^{ij} u^*_{i,j\pm} + {}^1/{}_2\psi_{\theta\theta\pm} T^2 + \psi^{ij}_{\theta\pm} u^*_{i,j\pm} T + \\ &\quad {}^1/{}_2\psi_\pm^{ijkl} u^*_{i,j\pm} u^*_{k,l\pm} + \ldots \\ p_\pm^{*ji} &= m_\pm(\psi_\pm^{ij} + \psi^{ij}_{\theta\pm} T + \psi_\pm^{ijkl} u^*_{k,l\pm} + {}^1/{}_2\psi^{ij}_{\theta\theta\pm} T^2 + \\ &\quad \psi^{ijkl}_{\theta\pm} u^*_{k,l\pm} T + {}^1/{}_2\psi_\pm^{ijklmn} u^*_{k,l\pm} u^*_{m,n\pm}) + \ldots \end{aligned} \tag{1.3}$$

Assuming that the phases are isotropic and the reference configurations are undistorted, using the relations given in /4/, we can write the expansion coefficients in the form

$$\begin{aligned} m_\pm\psi_\pm{}^{ij} &= -p_\pm{}^\circ x_\pm{}^{ij}, \quad m_\pm\psi^{ij}_{\theta\pm} = -K_\pm\alpha_\pm x_\pm{}^{ij} \\ m_\pm\psi_\pm^{ijkl} &= p_\pm{}^\circ(x_\pm{}^{il}x_\pm^{jk} - x_\pm{}^{ij}x_\pm^{kl}) + \lambda_\pm x_\pm{}^{ij}x_\pm^{kl} + \mu_\pm(x_\pm^{ik}x_\pm{}^{jl} + x_\pm{}^{il}x_\pm^{jk}) \end{aligned} \tag{1.4}$$

where $p_{\pm}^{\circ}$, $\lambda_{\pm}$, $\mu_{\pm}$, $K_{\pm}$, $\alpha_{\pm}$ are the pressure, isothermal Lamé, and the volume compression moduli, and the thermal expansion coefficient in the reference configurations of the phases at the temperature θ°.

2. Asymptotic form of the small density differences. We assume that both phases in question are isotropic, the reference configurations given above are undistorted, and that at the temperature θ° the initial pressures $p_{\pm}^{\circ}$ and the specific Gibb's potentials are the same

$$p_+^{\circ}=p_-^{\circ}=p, \quad \psi_+^{\circ}+p/m_+=\psi_-^{\circ}+p/m_- \tag{2.1}$$

We further assume that the similarity coefficient is nearly equal to unity

$$d=1+\delta\varepsilon;\quad \delta\sim 1,\quad \varepsilon\ll 1 \tag{2.2}$$

and the displacement fields at the outer boundary are of order ε.

In this situation it is natural to expect that an equilibrium configuration may exist, containing both phases, and that the parameters of both phases will differ little from the reference parameters, so that the phase displacement fields $\mathbf{u}_{\pm}$, the equation of the interphase boundary $x^i(\xi^{\alpha})$ and the increment in the equilibrium temperature T will all be represented in the form of series in the small parameter ε

$$\mathbf{u}_{\pm}=\sum_{N=1}^{\infty}\varepsilon^N\mathbf{u}_{N\pm},\quad x^i(\xi,\varepsilon)=\sum_{N=0}^{\infty}\varepsilon^N x^i_{\cdot N}(\xi),\quad T=\sum_{N=1}^{\infty}\varepsilon^N T_N. \tag{2.3}$$

Let us now derive the equations for determining the first non-zero terms of these expansions. We note that the functions $x^i_{\cdot N}(\xi)$ are defined to within an arbitrary coordinate substitution on the surface ξ^{α}. To localize this ambiguity in the function $x_0^i(\xi)$ we shall seek, in what follows, the equation of the surface in the form

$$x^i(\xi,\varepsilon)=x^i_{\cdot 0}(\xi)+n_0^i(\xi)\sum_{N=1}^{\infty}\varepsilon^N a_N(\xi) \tag{2.4}$$

where n_0^i are the components of the unit normal to the surface $x_0^i(\xi)$ and the series yields the distance from the surface $x^i(\xi,\varepsilon)$ to x_0^i, along the normal to the latter.

The following relations follow directly from geometrical considerations and expansions (2.2), (2.3):

$$\begin{aligned}
&x_{i-}=x_{i-}(1+\delta\varepsilon),\quad x_{ij-}=d^2x_{ij+}=(1+2\delta\varepsilon+\delta^2\varepsilon^2)\,x_{ij+},\\
&x_-^{ij}=(1-2\delta\varepsilon+3\delta^2\varepsilon^2)\,x_+^{ij}+o(\varepsilon^2),\quad u^*_{i-}=u_{i-}(1+\delta\varepsilon),\\
&U_-^i=u_-^i+\delta\varepsilon x^i,\quad u^i\equiv u\cdot x_+^i=\delta\varepsilon x^i\\
&m_-=m_+(1-\delta\varepsilon)^{-3}=m_+(1-3\delta\varepsilon+6\delta^2\varepsilon^2)+o(\varepsilon^2)
\end{aligned} \tag{2.5}$$

Combining relations (2.1), (2.5) we obtain

$$\psi_-^{\circ}=\psi_+^{\circ}+p\,(m_+^{-1}-m_-^{-1})=\psi_+^{\circ}-3p\delta\varepsilon\,(1+\delta\varepsilon)/m_++o(\varepsilon^2) \tag{2.6}$$

Combining further relations (1.3), (1.4), (2.1)–(2.6) one after the other, we obtain

$$\begin{aligned}
&\psi_+=\psi_+^{\circ}+\varepsilon\left(\psi_{\theta+}T_1-\frac{p}{m}v^i_{\cdot,i+}\right)+o(\varepsilon)\\
&p_+^{ji}=-px_-^{ij}+\varepsilon\,\{p\,(v^{j\,i}_{\cdot,\cdot+}-x_+^{ij}v^k_{\cdot,k+})-K_+\alpha_+T_1x_+^{ij}+\lambda_+x_+^{ij}v^k_{\cdot,k+}+2\mu_+v^{(i\,j)}_{\cdot,\cdot-}\}+o(\varepsilon)\\
&\psi_+x_+^{ij}-\frac{1}{m_+}p_+^{ik}u_{k,\cdot+}^{\cdot\,j}=\psi_+^{\circ}x_+^{ij}+\varepsilon\left\{\frac{p}{m_+}v^{i\,j}_{\cdot,\cdot+}+\left(\psi_{\theta+}T_1-\frac{p}{m_+}v^k_{\cdot,k+}\right)x_+^{ij}\right\}+o(\varepsilon)\\
&\psi_-=\psi_-^{\circ}+\varepsilon\left\{\psi_{\theta-}T_1-\frac{p}{m_+}(3\delta+v^k_{\cdot,k-})\right\}+o(\varepsilon)\\
&p_-^{ji}=d\,\frac{m_-}{m_-}p_-^{*ji}=-px_+^{ij}+\varepsilon\,\{p\,(v^{i\,i}_{\cdot,\cdot-}-x_+^{ij}v^k_{\cdot,k-}-2\delta x_+^{ij})-\\
&\qquad K_-\alpha_-T_1x_+^{ij}+\lambda_-x_+^{ij}v^k_{\cdot,k-}+2\mu_-v^{(i\,j)}_{\cdot,\cdot-}\}+o(\varepsilon)\\
&\psi_-x_-^{ij}-\frac{1}{m_+}p_-^{ik}(u_{k,\cdot-}^{\cdot\,j}+\varepsilon\delta\delta_k^{\ j})=\psi_+^{\circ}x_+^{ij}+\\
&\qquad \varepsilon\left\{\psi_{\theta-}T_1x^{ij}+\frac{p}{m_+}(v^{i\,j}_{\cdot,\cdot-}-x_+^{ij}v^k_{\cdot,k-}-2\delta x_+^{ij})\right\}+o(\varepsilon),\quad v^i\equiv u_1^i
\end{aligned} \tag{2.7}$$

Let us substitute relations (2.7) into (1.1). Equating the coefficients accompanying ε we obtain, respectively,

$$\begin{aligned}
&\text{a) } (\lambda x_+^{ij}v^k_{\cdot,k}+2\mu v^{(ij)}_{\cdot,\cdot})_{,j}=0,\quad \text{b) } [v^j]_-^{\,+}=\delta x_0^j\\
&\text{c) } [p\,(v^{j\,i}_{\cdot,\cdot}-x_+^{ij}v^k_{\cdot,k})-K\alpha T_1x_+^{ij}+\lambda x_+^{ij}v^k_{\cdot,k}+2\mu v^{(i\,j)}_{\cdot,\cdot}]_-^{\,+}n_{j0}=-2p\delta n^i_{\cdot 0}\\
&\text{d) } \left[\frac{p}{m}(v^{i\,j}_{\cdot,\cdot}-x_+^{ij}v^k_{\cdot,k})+\psi_{\theta}T_1x_+^{ij}\right]_-^{+}n_{i0}n_{j0}=-2p\delta
\end{aligned} \tag{2.8}$$

Here and henceforth n_{jM} will denote the coefficient of ε^M in the expansion of the components of the unit normal to the surface separating the phases. We can eliminate from the last two relations of (2.7) the terms containing the initial pressure p, using the relations

$$[v^j_{\cdot,i}]_-^+(x_+^{ik}-n_0{}^i n_0{}^k)=\delta(x_+^{jk}-n^j_{\cdot 0}n^k_{\cdot 0}),\quad [v^j_{\cdot,j}]_-^+= \tag{2.9}$$
$$[v^{j\,i}_{\cdot,\cdot}]_-^+ n_{j0}n_{i0}+2\delta,\quad [v^{j\,i}_{\cdot,\cdot}]_-^+ n_{j0}=([v^k_{\cdot,k}]_-^+-2\delta)\,n^i_{\cdot 0}$$

(in (2.8), (2.9) all discontinuities are calculated at the surface x^i_0).

To prove relations (2.9), we differentiate the second relation of (2.8) in ξ^α, convolute the result with $x^k_{\cdot\beta}\,\xi_0^{\alpha\beta}$ ($\xi_0^{\alpha\beta}$ is the metric tensor at the surface x^i_0) and use the well-known identity $x^i_{\cdot\alpha}x^k_{\cdot\beta}\xi_0^{\alpha\beta}=x_+{}^{ik}-n_0{}^i n_0{}^k$. This yields instantly the first of the relations under proof. Dropping from it the index k and convoluting the result over the indices j, k, we arrive at the second relation. Finally, convoluting the first relation of (2.9) with n_{j}, and using the second relation, we confirm the validity of the third relation.

Using (2.9) we reduce the last pair of relations (2.8) to the form

$$[\lambda x_+{}^{ij}v^k_{\cdot,k}+2\mu v^{(ij)}_{\cdot,\cdot}]_-^+ n_{j0}=0,\quad [\psi_\theta]_-^+T_1=0 \tag{2.10}$$

The second relation obtained leads to an important conclusion: when $[\psi_\theta]_-^+\neq 0$ (which represents the general situation in the case of phase tranasitions of the first kind when the latent heat of transformation is different from zero), then the temperature of phase equilibrium in a heterogeneous configuration is equal, to a first approximations, to the reference temperature θ° /5/. Therefore to determine the temperature effect we must determine T_2. This can be done by equating terms of the second order of smallness ε in the last relation of (1.1). Cumbersome calculations yield the formula

$$m_+[\psi]_-^+T_2+{}^1/_2\,[\lambda v^i_{\cdot,i}v^j_{\cdot,j}+2\mu v^{(ij)}_{\cdot,\cdot}v_{(i,j)}]_-^+-[\lambda v^k_{\cdot,k}v^{ij}_{\cdot\cdot}+2\mu v^{(ik)}_{\cdot,\cdot}v_{k,}{}^j]_-^+n_{i0}n_{j0}+\lambda_-\delta v^i_{\cdot,i-}+2\mu_-\delta v^{ij}_{\cdot,\cdot-}n_{i0}n_{j0} \tag{2.11}$$

This functional equation also plays a major role in determining the position of the unknown interphase boundary /5/. We shall therefore present here the key concepts and formulas necessary to determine it.

Let us differentiate the last equation of (1.1) covariantly twice with respect to the scalar parameter ε /6/ (naturally, after substituting the series (2.3)). Taking into account relations (2.1), the homogeneity of the reference configurations and the properties of the $\delta/\delta\varepsilon$ - derivative we obtain, for $\varepsilon=0$,

$$\left\{\frac{1}{2}\left[\frac{\partial^2 v^{ij}(x,\varepsilon)}{\partial\varepsilon^2}\right]_-^+ + cn^k\left[\frac{\partial v^{ij}}{\partial\varepsilon_{,k}}\right]_-^+\right\}n_in_j+\left[\frac{\partial v^{ij}}{\partial\varepsilon}\right]_-^+\frac{\delta n_in_j}{\delta\varepsilon}\bigg|_{\varepsilon=0}=0 \tag{2.12}$$

(c is the "velocity" of the boundary induced by varying the parameter ε).

Using the relations (1.2)–(1.4), (2.3), (2.5)–(2.7), (2.10) we obtain ($;\alpha$ is the symbol of covariant differentiation with respect to the coordinate ξ^α on the surface $x^i_0(\xi)$)

$$n_i\big|_{\varepsilon=0}=n_{i0},\quad c\big|_{\varepsilon=0}=c_1,\quad \frac{\delta n_i}{\delta\varepsilon}\bigg|_{\varepsilon=0}=-a_{1;\alpha}x^{\cdot\alpha}_{i\cdot 0} \tag{2.13}$$

From the relations (1.2)–(1.4), (2.3)–(2.7), (2.10) we obtain

$$\frac{\partial v^{ij}}{\partial\varepsilon_+}\bigg|_{\varepsilon=0}=\frac{p}{m_-}\,(v^{ij}_{\cdot,\cdot-}-x_-^{ij}v^k_{\cdot,k-}) \tag{2.14}$$
$$\frac{\partial v^{ij}}{\partial\varepsilon_-}\bigg|_{\varepsilon=0}=\frac{p}{m_+}\,(v^{ij}_{\cdot,\cdot-}-x_-^{ij}v^k_{\cdot,k-}-2\delta x_-^{ij})$$
$$\frac{\partial v^{ij}}{\partial\varepsilon}_{,k\pm}\bigg|_{\varepsilon=0}=\frac{p}{m_+}\,(v^{ij}_{\cdot,\cdot k\pm}-v^l_{\cdot,lk\pm}x_-^{ij})$$
$$\frac{1}{2}\frac{\partial^2v^{ij}}{\partial\varepsilon_+{}^2}\bigg|_{\varepsilon=0}n_{i0}n_{j0}=\chi_+m_-^{-1},\quad \frac{1}{2}\frac{\partial^2v^{ij}}{\partial\varepsilon_-{}^2}\bigg|_{\varepsilon=0}n_{i0}n_{j0}=$$
$$\chi_-m_-^{-1}-\delta^2-\delta v^k_{\cdot,k-}+\delta v^{ij}_{\cdot,\cdot-}n_{i0}n_{j0}-\frac{\lambda_-}{m_+}\delta v^i_{\cdot,i-}-\frac{2\mu_-}{m_+}\delta v^{ij}_{\cdot,\cdot-}n_{i0}n_{j0}$$
$$\chi_\pm\equiv m_+\psi_{\theta\pm}T_2+\frac{\lambda_\pm}{2}v^k_{\cdot,k\pm}v^l_{\cdot,l\pm}+\mu_\pm v_{(k,l)\pm}v^{kl}_{\cdot,\cdot\pm}-(\lambda_\pm v^k_{\cdot,k-}v^{ij}_{\cdot,\cdot\pm}+$$
$$2\mu_\pm v^{(ki)}_{\cdot,\cdot\pm}v_{k,}{}^j_{\cdot\pm})\,n_{i0}n_{j0}+p\,\{u^{ij}_{\cdot,\cdot 2\pm}n_{i0}n_{j0}-u^i_{\cdot,i2\pm}+\frac{1}{2}(v^i_{\cdot,j\pm}v^j_{\cdot,i\pm}-v^i_{\cdot,i\pm}v^j_{\cdot j\pm})-(v^{ik}_{\cdot,\cdot\pm}v_{k,}{}^j_{\cdot\pm}-v^k_{\cdot,k-}v^{ij}_{\cdot,\cdot-})\,n_{i0}n_{j0}\}$$

Using (2.13), (2.14) we write (2.12) in the form

$$[\psi_\theta]_-^+m_+T_2+\frac{1}{2}[\lambda v^i_{\cdot,i}v^j_{\cdot,j}+2\mu v^{(ij)}_{\cdot,\cdot}v_{i,j}]_-^+-[\lambda v^k_{\cdot,k}v^{ij}_{\cdot,\cdot}+2\mu v^{(ik)}_{\cdot,\cdot}v_{k,}{}^j]_-^+n_{i0}n_{j0}+\lambda_-\delta v^i_{\cdot,i-}+2\mu_-\delta v^{(ij)}_{\cdot,\cdot-}n_{i0}n_{j0}+ \tag{2.15}$$
$$p\,\{[u^{ij}_{\cdot,\cdot 2}]_-^+n_{i0}n_{j0}-[u^i_{\cdot,i2}]_-^++\frac{1}{2}[v^{ij}_{\cdot,\cdot}v_{j,i}-v^i_{\cdot,i}v^j_{\cdot,j}]_-^+-$$
$$[v^{ik}_{\cdot,\cdot}v_{k,}{}^j-v^{ij}_{\cdot,\cdot}v^k_{\cdot,k}]_-^+n_{i0}n_{j0}+\delta^2+\delta v^i_{\cdot,i-}-\delta v^{ij}_{\cdot,\cdot-}n_{i0}n_{j0}+a_1n_0{}^kn_{i0}n_{j0}\times$$
$$[v^{ij}_{\cdot,\cdot k}-v^l_{\cdot,lk}x_+{}^{ij}]_-^+-2a_{1;\alpha}x^{\cdot\alpha}_{(i\cdot 0}n_{j)0}([v^{ij}_{\cdot,\cdot}-v^k_{\cdot,k}x_+{}^{ij}]_-^++2\delta x_+{}^{ij})\}=0$$

(remembering that all discontinuities are calculated at the surface $x^i_0(\xi)$).

We note that when $p=0$ (2.15) instantly yields relation (2.11). However, to show the universal character of the latter we must show that the coefficient accompanying p in (2.15) vanishes. Substituting (2.2), (2.3) into (3.1) and equating the coefficients of ε^2, we obtain

$$[u^i{}_2]_-^+ + [v^i_{\cdot,j}]_-^+ n^j_{\cdot 0} a_1 = \delta n^i_{\cdot 0} a_1 \tag{2.16}$$

Let us now differentiate (2.16) in ξ^α and convolute the result with the tensor $x^k_{\cdot\beta 0}\xi^{\alpha\beta}_0$. Using the notation $b_{ij} = b_{\alpha\beta} x^{\cdot\alpha}_{i\cdot 0} x^{\cdot\beta}_{j\cdot 0} = -n_{i0;\,\alpha} x^{\cdot\alpha}_{j\cdot 0}$ where $b_{\alpha\beta}$ is the coefficient of the second quadratic form of the surface x^i_0, we obtain

$$[u^i_{\cdot,l2}]_-^+ (x_-^{kl} - n_0^k n_0^l) + [v^i_{\cdot,jl}]_-^+ (x_-^{kl} - n_0^k n_0^l) n^j_{\cdot 0} a_1 + [v^i_{\cdot,j}]_-^+ ({}_{1;\alpha} x^{k\alpha}_{\cdot\cdot 0} n_0^j - a_1 b^{jk}) = \delta (a_{1;\alpha} x^{k\alpha}_{\cdot\cdot} n^i_{\cdot 0} - b^{ki} a_1) \tag{2.17}$$

Neglecting the index k in (2.17) and convoluting with respect to i, k, we obtain a relation which can be reduced, using (2.9) and the identity $x^i_{\cdot\alpha 0} n_{i0} = 0$, to the form

$$[u^{i\,j}_{\cdot,\cdot 2}]_-^+ n_{i0} n_{j0} - [u^i_{\cdot,i2}]_-^+ - [v^i_{\cdot,j}]_-^+ a_{1;\,\alpha} x^{\cdot\alpha}_{i\cdot 0} n^j_{\cdot 0} + a_1 n_{i0} n_0^j n_0^l [v^i_{\cdot,jl} - v^{i\,k}_{\cdot,\cdot k} x_{jl+}]_-^+ = 0 \tag{2.18}$$

Now we can confirm that the coefficient of p in (2.15) vanishes, using relations (2.9), (2.18) and

$$1/2\, [v^{i\,j}_{\cdot,\cdot} v_{j,\,i} - v^i_{\cdot,i} v^j_{\cdot,j}]_-^+ - [v^{ik}_{\cdot,\cdot} v_{k,\cdot}^{\cdot\,j} - v^k_{\cdot,k} v^{i\,j}_{\cdot,\cdot}]_-^+ n_{i0} n_{j0} = -\delta^2 - \delta v^i_{\cdot,i-} + \delta v^{i\,j}_{\cdot,\cdot -} n_{i0} n_{j0} \tag{2.19}$$

Relation (2.19) can be obtained from the following four relations:

$$\begin{aligned}
&[v^i_{\cdot,i} v^j_{\cdot,j}]_-^+ = 2v^i_{\cdot,i-} (h^l n_{l_0} + 2\delta) + (h^l n_{l_0} + 2\delta)^2 \\
&[v^i_{\cdot,j} v^j_{\cdot,i}]_-^+ = 2v^i_{\cdot,j-} h^j n_{i0} + 2\delta (v^i_{\cdot,i-} - v^{i\,j}_{\cdot,\cdot -} n_{i0} n_{j0} + \delta) - (h^l n_{l_0})^2 \\
&[v^{ik}_{\cdot,\cdot} v_{k,\cdot}^{\cdot\,j}]_-^+ n_{i0} n_{j0} = v^{ki}_{\cdot,\cdot} h_i n_{k0} + v^{i\,j}_{\cdot,\cdot -} n_{i0} n_{j0} h^k n_{k_0} + (h^i n_{i0})^2 \\
&[v^k_{\cdot,k} v^{i\,j}_{\cdot,\cdot}]_-^+ n_{i0} n_{j0} = v^{i\,j}_{\cdot,\cdot -} n_{i0} n_{j0} (h^k n_{k0} + 2\delta) + h^l n_{l_0} (v^k_{\cdot,k-} + h^k n_{k_0} + 2\delta)
\end{aligned} \tag{2.20}$$

Here $h^i \equiv [v^i_{\cdot,j}]_-^+ n^j_{l_0}$. To establish relations (2.20) we must use (2.9) and the following expression for the discontinuity of the product:

$$[ab]_-^+ = a_- [b]_-^+ + b_- [a]_-^+ + [a]_-^+ [b]_-^+$$

Melting, regarded as a transformation of the solid phase into the liquid phase, can be referred to a number of phase transitions with slippage, for which the asymptotic form of small density differences was developed in /5/* (*See also: Grinfel'd M.A. Heterogeneous systems with phase transition surfaces (application of variational principles. Doctorate Dissertation, Institute of Terrestrial Physics, Moscow, 1983).

It can however be shown, that in the case when one of the phases is liquid, the relations (2.8, a, b), (2.10), (2.11) lead to equations obtained for the case of melting in the papers mentioned.

Indeed, let the minus phase be liquid ($\mu_- = 0$) and occupy the volume V within the plus phase in the equilibrium configuration. In this case the first equation of (2.8) for the minus phase yields

$$v^k_{\cdot,k-} = H = \text{const}, \quad x \in V \tag{2.21}$$

The first relation of (2.10) can now be written in the form

$$(\lambda_+ v^k_{\cdot,k-} x_+^{ij} - 2\mu_- v^{(i\,j)}_{\cdot,\cdot -}) n_{j_0} = K_- H n^i_{\cdot 0} \tag{2.22}$$

Integrating Eq.(2.21) over the finite volume V and using the relation (2.8c), we obtain

$$\begin{aligned}
HV &= \int_V dV v^k_{\cdot,k-} = \int_\xi d\xi v^k_- n_{k0} = \int_\xi d\xi (v^k_- - \delta x^k) n_{k0} = \\
&\int_\xi d\xi v^k_- n_{k0} - \Delta V n_-^{-1} - \int_\xi d\xi v_-^k n_{k_0} = V\left(H + \frac{\Delta}{m_+}\right) \quad (\Delta \equiv 3\delta m_+)
\end{aligned} \tag{2.23}$$

Further, using the last relation of (2.9) and first relation of (2.10), as well as $\mu_- = 0$, we obtain

$$[\lambda v^k_{\cdot,k} v^{i\,j}_{\cdot,\cdot} + 2\mu v^{(i\,k)}_{\cdot,\cdot} v_{k,\cdot}^{\cdot\,j}]_-^+ n_{i0} n_{j0} = (\lambda v^l_{\cdot,l} x_-^{ki} + 2\mu v^{(k\,i)}_{\cdot,\cdot})_\pm n_{i0} n_{j0} [v_{k,\cdot}^{\cdot\,j}]_-^+ = K_- v^l_{\cdot,l-} ([v^i_{\cdot,i}]_-^+ - 2\delta) \tag{2.24}$$

Now, using (2.24) we reduce (2.11) to the form

$$2m_+ [\psi]_-^+ T_2 + \lambda_+ v^i_{\cdot,i-} v^j_{\cdot,j-} + 2\mu_+ v^{(i\,j)}_{\cdot,\cdot -} v_{i,\,j+} + K_- H^2 - 2K_- H (v^l_{\cdot,l+} - \Delta/m_+) = 0 \tag{2.25}$$

This completes all conditions of equilibrium of /5/.

3. Determination of the parameters of the ellipsoidal equilibrium inclusions of the solid phase. We shall consider, in the small denisty difference approximation, the problem of bounded equilibrium inclusions of the solid phase for the case of coherent transformations and the displacement field u^i_+ linear at infinity (in the case of phase transitions with slippage the problem was discussed in /5, 7/**(**See the previous footnote)

$$\lim_{x^i \to \infty} v_+{}^i(x) = \varkappa^i_{\cdot j} x^j \tag{3.1}$$

We shall seek an equilibrium inclusion in the form of a triaxial ellipsoid, and the field $v^i_\pm$ in the form (compare with /7/)

$$v_{i-}=\beta_{ij}x^j, \quad v_{i+}=\frac{\gamma}{4\pi}\varphi_{,i}+\varkappa_{ij}x^j \tag{3.2}$$

where φ is the Newtonian potential of the corresponding ellipsoid of unit density, and β_{ij} and γ are constants. The inner potential of the homogeneous ellipsoid with the centre at the origin of coordinates is a quadratic form

$$\varphi=\varphi_0-{}^1/_2\omega_{ij}x^ix^j, \quad \varphi_0, \omega_{ij} - \text{const} \tag{3.3}$$

Its coefficients are defined by the oriented and form of the ellipsoid. Knowing them, we can solve the inverse problem and find the orientation and magnitudes of the excentricities.

Inside and outside the body the potential satisfies the equations

$$\text{a) } \varphi_{,\cdot i}^{\cdot i}=-4\pi, \quad \text{b) } \varphi_{,\cdot i}^{\cdot i}=0 \tag{3.4}$$

so that $\omega_{\cdot i}^{i}=4\pi$. Moreover, the potential φ vanishes at infinity (together with all derivatives), is continuous at the boundary Σ of the ellipsoid together with the first derivatives, and the discontinuities in the second derivatives at the boundary Σ satisfy the compatibility relations $[\varphi_{,ij}]_-^+=4\pi N_iN_j$ (N_i is the unit normal to the ellipsoid surface). It can be shown that the functions $v_{i\pm}$ defined by relations (3.2) satisfy the equations of equilbrium and the boundary condition at infinity. Combining the compatibility relations with (3.2), (3.3), we obtain

$$v_{i,j-}|_\Sigma=\gamma\left(N_iN_j-\frac{1}{4\pi}\omega_{ij}\right)+\varkappa_{ij} \tag{3.5}$$

Using (3.1)–(3.5) we can write the second boundary condition of (2.8) and the first condition of (2.10) as linear forms of the coordinates x^i and as components of the unit normal N_i respectively. Since the choice of these quantities on the ellipsoid surface is arbitrary, we conclude that the necessary condition for the solution of the type shown to exist is, that the following relations connecting the constants hold:

$$\frac{\gamma}{4\pi}\omega^{ij}-\varkappa^{ij}+\beta^{ij}+\delta x^{ij}=0 \tag{3.6}$$

$$\lambda_+\varkappa_{\cdot k}^{k}x^{ij}+2\mu_+\varkappa^{(ij)}+2\mu_+\gamma\left(x^{ij}-\frac{\omega^{ij}}{4\pi}\right)-\lambda_-\beta_{\cdot k}^{k}x^{ij}-2\mu_-\beta^{(ij)}=0 \tag{3.7}$$

We shall consider the relations (3.4), (3.6), (3.7) as a system of equations in ω^{ij}, β^{ij}, γ. From (3.6) it follows that ($[ij]$ is the symbol of alternation)

$$\beta^{[ij]}=\varkappa^{[ij]} \tag{3.8}$$

Symmetrizing system (3.6), multiplying it by $2\mu_+$ and combining the resulting expression with (3.7), we obtain

$$\beta^{(ij)}=\frac{\lambda_+\varkappa_{\cdot k}^{k}-\lambda_-\beta_{\cdot k}^{k}-2\mu_+(\gamma+\delta)}{2(\mu_--\mu_+)}=\beta x^{ij} \tag{3.9}$$

Combining (3.6) and (3.9) we have

$$\omega^{ij}=\frac{4\pi}{\gamma}[\varkappa^{(ij)}-(\beta+\gamma)x^{ij})] \tag{3.10}$$

It remains to determine the constants γ and β. To do this, we convolute (3.9), (3.10) over the free indices. Using (3.4) we obtain a system whose solution yields

$$\gamma=\frac{(K_--K_+)\varkappa_{\cdot k}^{k}-3K_-\delta}{K_-+{}^4/_3\mu_+}, \quad \beta=\frac{(K_++{}^4/_3\mu_+)\varkappa_{\cdot k}^{k}-4\mu_+\delta}{3K_-+4\mu_+} \tag{3.11}$$

In order for the functions (3.2) to provide a solution to the problem of phase equilibrium, it is also necessary that the relation (2.11) should hold at all points of the ellipsoid surface. Using the relations (3.2), (3.5), (3.8)–(3.11) we confirm that the latter requirement will be satisfied, provided that the temperature T_2 is given by the formula (σ^{ij} denote the stresses at infinity)

$$\begin{aligned}-2m_+[\psi_\theta]_-^+T_2 &= \frac{\lambda_+}{9K_+^2}(\sigma_{\cdot k}^{k})^2+\frac{1}{2\mu_+}\left[3\left(K_-H-2\mu_+\gamma-\frac{\lambda_+}{3K_+}\sigma_{\cdot k}^{k}\right)^2+\right. \\ &\left.4\mu_+\gamma\left(K_-H-\mu_+\gamma-\frac{\lambda_+}{3K_+}\sigma_{\cdot k}^{k}\right)\right]+K_-H^2- \\ &2K_-H\left(\frac{1}{3K_+}\sigma_{\cdot k}^{k}-\frac{\Delta}{m_+}\right)\end{aligned} \tag{3.12}$$

$$\sigma^{ij}\equiv\lambda_+\varkappa_{\cdot k}^{k}x^{ij}+2\mu_+\varkappa^{(ij)}$$

The solution shows that in the case in question the stress in the ellipsoidal including is hydrostatic, and the equilibrium temperature, the form of the ellipsoid and the stress state outside it, are all described by the same relations as in the case of the transition of an isotropic solid to the liquid state. In this connection numerous other relations obtained in the papers cited also remain valid.

We note that the posssibility of constructing solutions with the inclusion of a new phase in the form of an ellipsoid, is connected only with the condition of coherence, the isotropic character of the matrix, the possibility of approximating the free energy density of the matrix by expanding the second-order infinitesimals in the displacement gradients, and the constancy of these gradients at infinity. At the same time, the assumption of the isotropic character and the linear elasticity of the inclusion, and of the special character of the affine "characteristic" deformation Δ_{ij}, does not in any way represent a significant restriction when constructing solutions of the type shown. In particular, in the case of isotropic phases but of an arbitrary intrinsic deformation tensor, the relations obtained in /8/ can be used to find the following formulas generalizing (3.9), (3.10):

$$
\begin{aligned}
&\beta_{(ij)} = x_{ij}L - \frac{\mu_+}{\mu_+ - \mu_-}\Delta_{(ij)} \\
&\beta_{[ij]} = \varkappa_{[ij]} - \Delta_{[ij]} \\
&\frac{\omega^{ij}}{4\pi}\gamma^* = Q^{ij} - x^{ij}L \\
&L = \frac{\lambda_- + 2\mu_-}{3K_- + 4\mu_+}\left(\frac{\lambda_+ + 2\mu_+}{\lambda_- + 2\mu_-}\varkappa^{k}_{\cdot k} + \frac{\mu_+}{\mu_+ - \mu_-}\Delta^{k}_{\cdot k}\right) \\
&Q^{ij} = \varkappa^{(ij)} + \frac{\mu_-}{\mu_+ - \mu_-}\Delta^{(ij)} \\
&\gamma^* = \frac{(K_- - K_+)\varkappa^{k}_{\cdot k} - K_-\Delta^{k}_{\cdot k}}{K_- + {}^4/_3\mu_+}
\end{aligned}
\tag{3.13}
$$

From (3.13) it follows that the ellipsoid is coaxial with the tensor Q^{ij}, and the latter, in turn, is coaxial with the stress tensor at infinity in the case of melting $(\mu_- = 0)$, or in the case of intrinsic tensile volume deformation $\Delta_{ij} = \delta x_{ij}$. In the case of volume expansion at infinity the ellipsoid is coaxial with the intrinsic deformation tensor $\Delta_{(ij)}$. In the remaining cases the orientation of the ellipsoid is governed by both factors, by the character of the deformations at infinity, and by the intrinsic deformation of the transformation.

4. Heterogeneous configuration with homogeneous stress-strain states of the phases. First we use the relations of Sect.2 as the basis for investigating the problem of equilibrium coexistence of the half-spaces composed of different isotropic elastic phases subjected to affine deformation. Thus, let the plus(minus) phase be subjected to affine deformation $v_i^+ = \varkappa^{i}_{\cdot j+}x^j$ $(v^i_- = \varkappa^{i}_{\cdot j-}x^j)$ relative to its reference configuration, where $\varkappa^{i}_{\cdot j\pm}$ are constant tensors. If the above-spaces are in the state of complete thermodynamic equilibrium along the plane $b_i x^i = b_0$ (we can assume without loss of generality that the vector b_i is normalized to unity and therefore coincides with the unit normal to the plane n_i), then by virtue of the conditions of equilibrium (2.8), (2.10), (2.11) the following algebraic relations must hold (the equations of equilibrium within the phases are in this case satisfied automatically):

$$
\begin{aligned}
&[\varkappa^{i}_{\cdot j}]_-^+ x^j - \delta x^i |_{b_i x^i = b_0} = 0 \\
&[\lambda\varkappa^{k}_{\cdot k}x^{ij} + 2\mu\varkappa^{(ij)}]_-^+ n_i = 0 \\
&m_+[\psi_\theta]_-^+ T_2 + {}^1/_2[\lambda\varkappa^{i}_{\cdot i}\varkappa^{j}_{\cdot j} + 2\mu\varkappa^{(ij)}\varkappa_{ji}]_-^+ - [\lambda\varkappa^{k}_{\cdot k}\varkappa^{ij} + \\
&\quad 2\mu\varkappa^{(ik)}\varkappa^{\cdot j}_{k\cdot}]_-^+ n_i n_j + \lambda_-\delta\varkappa^{i}_{\cdot i-} + 2\mu_-\delta\varkappa_-^{ij}n_i n_j = 0
\end{aligned}
\tag{4.1}
$$

Differentiating the first of these relations with respect to the coordinates in the plane ξ^α and convoluting the result with $x_{k\cdot}^{\cdot\alpha}$, we obtain

$$
[\varkappa_{ik}]_-^+ = h_i n_k + \delta(x_{ik} - n_i n_k) \quad (h_i \equiv [\varkappa_{ij}]_-^+ n^j)
\tag{4.2}
$$

Only six of the nine relations of (4.2) are independent, since (4.2) becomes an identity when convoluted with the vector n^k. Consequently, the ten independent relations appearing in the last pair of system (4.1) and (4.2) connect 18 constants $\varkappa^{ij}_{\pm}$, two independent components of the unit normal n_i, and the increment in the equilibrium temperature T_2. Thus, to determine uniquely the piecewise homogeneous equilibrium configuration of the half-spaces in question, we can specify arbitrarily e.g. the affine deformation of the plus phase and the orientation of the boundary of separation n_i.

To determine to conjugate equilibrium stress-strain state of the minus phase (i.e. the tensor $\varkappa^{i}_{\cdot j-}$), we use (4.2) to represent the second relation of (4.1) in the form

$$
\{[\lambda]_-^+\varkappa^{k}_{\cdot k-}x^{ij} + 2[\mu]_-^+\varkappa^{(ij)}_{+} + \lambda_- h^k n_k x^{ij} + 2\mu_- h^{(i}n^{j)} + 2\delta\lambda_- x^{ij} + 2\delta\mu_-(x^{ij} - n^i n^j)\}\, n_j = 0
\tag{4.3}
$$

Convoluting (4.3) with n_i we obtain

$$h^k n_k = -\frac{1}{\lambda_- + 2\mu_-}\left([\lambda]_-^+ \varkappa^k_{\cdot k+} + 2[\mu]_-^+ \varkappa_+^{(kl)} n_k n_l + 2\delta\lambda_-\right) \tag{4.4}$$

Substituting (4.4) into (4.3) we obtain

$$h^i = n^i \frac{\lambda_- + \mu_-}{\mu_-(\lambda_- + 2\mu_-)}\left(-\frac{\mu_-[\lambda]_-^+}{\lambda_- + \mu_-}\varkappa^k_{\cdot k+} + 2[\mu]_-^+ \varkappa_+^{(kl)} n_k n_l - \right. \tag{4.5}$$

$$\left.\frac{2\delta\lambda_-\mu_-}{\lambda_- + \mu_-}\right) - \frac{2[\mu]_-^+}{\mu_-}\varkappa_+^{(ik)} n_k = n^i R - \frac{[\mu]_-^+}{\mu_+\mu_-}\sigma_+{}^{ij} n_j$$

$$R \equiv \frac{\mu_+\lambda_- - \lambda_+\mu_-}{\mu_+(\lambda_- + 2\mu_-)(3\lambda_+ + 2\mu_+)}\sigma^k_{\cdot k+} - \frac{2\delta\lambda_-}{\lambda_- + 2\mu_-} +$$

$$\frac{[\mu]_-^+(\lambda_- + \mu_-)}{\mu_-\mu_+(\lambda_- + 2\mu_-)}\sigma_+^{kl} n_k n_l$$

$$(\sigma_\pm{}^{ij} = \lambda_\pm \varkappa^k_{\cdot k\pm} x^{ij} + 2\mu_\pm \varkappa_\pm^{(ij)})$$

Here $\sigma_+{}^{ij}$ is the stress tensor in the plus phase. In deriving the expression for the vector h^i in terms of $\sigma_+{}^{ij}$, we used the relation

$$\varkappa_-^{(ij)} = \frac{1}{2\mu_+}\left(\sigma_+{}^{ij} - \frac{\lambda_+\sigma^k_{\cdot k+}}{3\lambda_+ + 2\mu_+} x^{ij}\right)$$

The tensor $\varkappa_-{}^{ij}$ and the stress tensor of the minus phase $\sigma_-{}^{ij}$ are given by

$$\varkappa_{ij-} = \varkappa_{ij+} - h_i n_j - \delta(x_{ij} - n_i n_j) = \tag{4.6}$$

$$\varkappa_{ij-} - n_i n_j \left(-\frac{\mu_-[\lambda]_-^+}{\lambda_- + \mu_-}\varkappa^k_{\cdot k-} + 2[\mu]_-^+ \varkappa_+^{kl} n_k n_l - \right.$$

$$\left.\frac{\mu_-(3\lambda_- + 2\mu_-)\delta}{\lambda_- + \mu_-}\right)\frac{\lambda_- + \mu_-}{\mu_-(\lambda_- + 2\mu_-)} - \delta x_{ij} + \frac{2[\mu]_-^+}{\mu_-}\varkappa_{(ik)+} n^k n_j$$

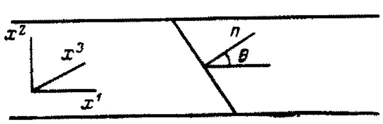
$$\sigma_-{}^{ij} = \lambda_- x^{ij}(\varkappa^k_{\cdot k+} - h^k n_k - 2\delta) + 2\mu_-\{\varkappa_+^{(ij)} - h^{(i}n^{j)} - \tag{4.7}$$

$$\delta(x^{ij} - n^i n^j)\} = 2\mu_- x^{ij}\left(R - \delta - \frac{[\mu]_-^+}{2\mu_-\mu_+}\right) - 2\mu_- n^i n^j (R - \delta) +$$

$$\frac{\mu_-}{\mu_+}\sigma_+{}^{ij} + \frac{[\mu]_-^+}{\mu_+} n_k (\sigma_+^{ik} n^j + \sigma_+^{jk} n^i)$$

The equilibrium temperature can now be found by substituting the formulas obtained into the last relation of (4.1).

The formulas of Sect.4 can be used to solve some boundary value problems. Let us consider, as an example, the two-dimensional problem of heterogeneous two-phase equilibrium in a strip loaded along the external boundaries $x^2 = \text{const}$ by a constant normal force $\sigma^{22} = -p$ and tangential force $\sigma^{21} = \tau$, with free surfaces $x^3 = \text{const}$ (see the figure). We shall consider the solutions for which there are corresponding affine deformations of both phases separated by the planes. Let us denote by θ the angle of inclination of the separating plane and by $\sigma_\pm$ the values of the stresses $\sigma^{11}{}_\pm$ in both phases. Using the condition of homogeneity of the phases, we conclude that the stress tensors in the phases have the form $(a, b = 1, 2)$



$$\|\sigma_\pm^{ab}\| = \begin{Vmatrix} \sigma_\pm & \tau \\ \tau & -p \end{Vmatrix} \tag{4.8}$$

The relations (4.7), (4.8) lead to the following equation for determining the quantities σ_+ and θ:

$$pc + \frac{2\mu_-(\mu_+\lambda_- - \lambda_+\mu_-)}{\mu_+(3\lambda_+ + 2\mu_+)(\lambda_- + 2\mu_-)}(\sigma_+ - p) + \tag{4.9}$$

$$\frac{\lambda_- cq}{\lambda_- + 2\mu_-} - \frac{2\delta\mu_-(3\lambda_- + 2\mu_-)}{\lambda_- + \mu_-} - R\sin^2\theta = 0$$

$$(c\sigma_+ - cp - R')\sin\theta\cos\theta = 0$$

Here we have used the notation

$$c = \frac{\mu_+ - \mu_-}{\mu_+}, \quad q = \sigma_+\cos^2\theta - 2\tau\sin\theta\cos\theta - p\sin^2\theta, \tag{4.10}$$

$$R' = \frac{2\mu_-}{\lambda_- + 2\mu_-}\left\{\frac{\mu_+\lambda_- - \lambda_+\mu_-}{\mu_+(3\lambda_+ + 2\mu_+)}(\sigma_+ - p) - (3\lambda_- + 2\mu_-)\delta + cq\frac{\lambda_- + \mu_-}{\mu_-}\right\}$$

Using (4.10), we can write the conditions of zero load on the surfaces $x^3 = \text{const}$ in the form

$$\sigma_-{}^{33} = R - cq = 0 \tag{4.11}$$

When $p = \tau = 0$, conditions (4.9), (4.11) take the form

$$D\cos^2\theta = 0,\ D\sin\theta\cos\theta = 0 \tag{4.12}$$
$$D + \frac{c(\lambda_- + 2\mu_-)}{6\mu_- K_-}\sigma_+\sin^2\theta = 0$$
$$D \equiv -\frac{1}{6}\left[\frac{\lambda}{\mu K}\right]_-^+\sigma_+ - \frac{c(\lambda_- + \mu_-)}{3\mu_- K_-}\sigma_+\sin^2\theta - \delta$$

System (4.12) has the following solutions:

$$1)\ \sin\theta = 0,\quad \sigma_+ = -6\delta \Big/ \left[\frac{\lambda}{\mu K}\right]_-^+ \quad (= \sigma_-)$$
$$2)\ \cos\theta = 0,\quad \sigma_+ = -6\mu_+\delta\left(\frac{\lambda_+}{K_+} + \frac{\lambda_-}{K_-}\right)^{-1} \quad \left(= \frac{\mu_+}{\mu_-}\sigma_-\right)$$

Thus we have for the first(second) solution the corresponding interphase planes perpendicular (parallel) to the z^1 axis.

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CONFIGURATIONAL FORCES IN THE MECHANICS OF A SOLID DEFORMABLE BODY*

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A configurational force /1—3/, which always originates in a deformable solid whenever the stress source moves, represents physically the contribution of the external strain and stress fields to the dissipation of energy, taken per unit path length of the source. When the stress source (singularity) is internal, the configurational force is the fundamental parameter controlling the process of motion and it can be called a driving force. Linear singularities of the type of crack and dislocation contours, point singularities of the type of small cavities and inclusions, etc. are examples of each cases. If the singularity is generated directly by external forces, the configurational force plays an auxiliary role and such cases will be examined below. This is the problem of the motion of a small solid body over the surface of a half-space, and different schemes of wedge motion in an unbounded elasto-plastic space.

1. Motion of a small solid over the surface of a half-space. Let a concentrated force $(T, 0, -N)$ move at a constant velocity V over the surface of a solid half-space $z < 0$ (Fig.1), stretched by a stress σ_x^∞. Its surface is considered to be free of external loads, with the exception of the point O moving with the velocity V of the origin. Since the field of quasistatic stresses and strains in a solid is stationary in the $Oxyz$ coordinate system, the following equality /1—3/ holds for any materials for any finite deformations:

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